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The Notation of Equal Temperaments

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I. INTRODUCTION

Interest in the theory of tunings other than 12-tone equal temperament (henceforth 12-tET) has increased dramatically in the twentieth century, largely through the pioneering music of composers such as Alois Hába, Julián Carrillo, Ivan Vyshnegradskiy, and Harry Partch. Readers of *Xenharmonikôn* likely know that aside from Partch, most microtonal composers before the Second World War investigated equal temperaments which are subdivisions of 12-tET, e.g. 24-, 36-, and 96-tET. Gardner Read (1990) has shown indirectly that much of the effort on notation of ETs has been driven by just such temperaments. For these, many composers have independently created simple but diverse notations, in many cases unrelated to their structural use of the various temperaments.

Despite the considerable history of ETs, few theorists have studied their properties or created musical notations for them systematically. Because most of the *ad hoc* solutions in notation of one ET do not extend easily to many other ETs, such solutions obscure similarities in related temperaments. Indeed, the questions of how equal temperaments are related and how they should be notated are twin aspects of the same issue. A few people have recently discussed either many ETs or ETs in general (e.g. Mandelbaum 1961; Regener 1973; Blackwood 1985, 1991; Rasch 1985). But despite some treatments in depth, we rarely find ways of interrelating ETs in terms of their basic generating intervals, nor notations which reflect these relations and help to explain them. Blackwood's and Rasch's notations possess the broadest implications. Despite problems found in them, the present article would not exist without the precedent of their work.

This article explores almost exclusively ETs of the octave and implications based on harmonics no higher than 5. However, the principles presented here are expandable to include higher harmonics and some systems which subdivide intervals other than the octave.

That ETs have not been used more, especially in functionally harmonic ways, despite the easy ability of many current synthesizers to produce the required intervals with tolerable resolution, may be because playing them on a keyboard of the familiar design is nearly impossible.[1] It may also be because notation of some ETs is by no means obvious. In any case,

[1] The most substantial non-improvised recorded project in different equal temperaments which I am aware of used, with considerable difficulty, a retunable synthesizer with two piano-like manuals for composition, and one with one manual for recording (Blackwood 1980). The *generalized keyboard,* laid out in two dimensions instead of the one found on the piano, is much better suited to many diverse tunings. Two instruments having such a keyboard, the Archiphone (fixed tuning) and some versions of the Scalatron (variable tuning), built in the 1970s in the Netherlands and the USA respectively, did not achieve widespread dissemination. The two-manual synthesizer which Blackwood used was an early version of the Scalatron; no generalized-keyboard Scalatron was available to him.

the traditional uses of notation need not disappear with the expansion of tonal resources which microtonality supplies.

II. BASIC PRINCIPLES

The presumption behind the discoveries and proposals in this article is centric music along the lines of the modal and tonal traditions of western music. "Along the lines" may be taken literally, for the notations about to be suggested also use the five-line staves of this music, the usual clefs, etc. While reform of this system is possible, especially in conjunction with microtonality, it is not the aim here. One aim is to provide notations which make as much use as practicable of the *system* which western musicians know. Because of this, new notations will be understandable and usable with a minimum of adjustment to prior knowledge. At the same time, certain details of common-practice notation must be explored and actually exploded, since most equal temperaments have little in common with 12-tET or the various tunings which were part of its background. The intent of this article is to provide notations of sounding pitches represented as part of a score as a musical object irrespective of performing medium or method.[2]

These presumptions lead first to the retention of the basic seven note names from A to G, with single or multiple sharps and flats. (H may be included for notations derived from German nomenclature.) The notes are modified by new accidental signs, most of which are derived from small intervals in just intonation. The reason why just intonation is the source of most signs should become clear as the discussion proceeds. The following points are offered as guidelines for the new signs. Each point will be discussed and illustrated, and a few more will be added later.

1. Additional signs must be perceptually distinct from each other, from the staff, from notes, and from other signs. They must be immediately attributable to specific notes, i.e. notes on specific lines or spaces.

2. Notation must reflect the determined nature of the temperament. If there is more than one way to derive or structure the temperament in terms of the relations among its intervals, there may be more than one notation for it.

3. Additional signs, beyond ♯ ♭ 𝄪 𝄫 etc., should be created for kommata representing the differences in pairs of multiples of certain basic just intervals.

Point 1 has been ignored surprisingly often. The commonly used + and – signs are difficult to position in front of notes because staff lines interfere with them. The sense of point 2 will

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[2]Such notation eschews the transpositions used for most scordatura, and also the tablature approach. A good example of scordatura in microtonality is a keyboard notation which indicates the keys to press but not what the pitches are that will sound. Many compositions for retuned synthesizer are written this way in order to avoid major problems for the performer; but such scores cannot be read musically as written, nor do they often use simple transpositions.

The entire issue of notation may be considered chimerical in the context of current electronic/computer music possibilities. Indeed, many such works are created without a score. This article does not address the issue of the utility of scores.

emerge shortly, because where we designate the perfect fifth in a temperament and how we consider the various kommata (see below) affect the whole use of the temperament. These differences should be reflected in differences of notation.

Point 3 is the heart of the matter: we need to consider only a few kommata to structure and hence notate nearly any equal temperament, regardless of how odd it is. These kommata and their derivation and size are contained in Chart 1. In it, a is the octave, υ the just perfect fifth, t the just major third, and θ the Pythagorean major third. By definition, $\theta = 4\upsilon - 2a$.

name	symbol	size (intervals)	also equals	size (cents)	M3s $\cong$ P8
syntonic komma	k	$\theta - t$	$4\upsilon - t - 2a$	21.506	*not applicable*
Pythagorean komma	p	$3\theta - a$	$12\upsilon - 7a$	23.460	3 Pythagorean
diesis	d	$a - 3t$	$3k - p$	41.059	3 just
skhisma	s	$2\theta + t - a$	$p - k$	1.954	2 Pyth., 1 just
diaskhisma	q	$a - (\theta + 2t)$	$2k - p$	19.553	1 Pyth., 2 just

CHART 1

All these kommata are well known in tuning theory. The last four of the five are the only ones necessary to capture the difference between a perfect octave (P8) and three major thirds (M3s) in combinations of Pythagorean and just, as the rightmost column shows. The syntonic komma does not arise from such a difference but from the more basic $\theta - t$.

Another way to characterize these kommata is by their involvement with t. All have t in their definition except the Pythagorean.[3] Because of this, p (the Pythagorean komma) is not required directly in considering the main problem in equal temperaments: to reconcile multiples of υ with multiples of t.

In choosing kommata with which to structure a temperament, we find that we usually do not need all of them in every temperament. One reason is that they are interdependent, as the formulas in the two columns of definitions show. Another is that in many temperaments, some of these kommata are not expressed, i.e. they are zero units in size. What happens when a komma is a negative number of units is deferred for later consideration.

In all temperaments, the single sharp and flat are given their usual Pythagorean meaning of a pitch class 7 steps along the line of perfect fifths. E.g. G♯ is defined as 7 υ's higher than G, regardless of the size of υ.

In ETs, no interval is just, i.e. represented by the ratios of just tunings, except for a, which is represented by the ratio 1:2. In ETs, therefore, the abbreviations υ, t, and θ imply *structurally just:* each denotes the number of units (abbr. *u.*) in the temperament which represents the respective just interval in the sense of coming close to it. E.g. "in 12-tET, υ = 7 u." means "in 12-tone equal temperament, the perfect fifth is represented by 7 units."

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[3]The Pythagorean komma may of course be defined as $3t + 3k - a$, but so doing defines θ in terms of k. Since we just defined k in terms of θ for the syntonic komma, we avoid this circularity by treating p as more basic than k—which it is, since p may involve only υ but k involves both υ and t.

Despite being less useful to notation, p provides the most basic categorization for equal temperaments. This may usually be determined by the formula $p = 5a \pmod{12}$ or $p = -7a \pmod{12}$.[4] Thus for 12-tET, $p = 0$, meaning that $12v - 7a = 0$, and B♯ and C are the same, since 12 perfect fifths above C is B♯. For 22-tET, however, $p = 2$, causing B♯ to lie 2 units higher than C; for 19-tET, $p = -1$, causing B♯ to lie 1 unit lower than C. It may be shown (Rapoport 1993) that for some large a's, p may take on several values differing by 12. The best p, defined as giving the value of v closest to just, may even have an absolute value greater than 12. But for many a's, the best p falls within the range –5 to +6, so that $-5 \leq p \leq +6$.

Similarly, for many a's, the best d, defined as giving the value of t closest to just, may be expressed by $0 \leq d \leq 2$. However, there are many good d's lying outside this range, because the possible d's for any a differ only by 3—not by 12, as in the possible p's. In other words, possible t's are closer to each other than possible v's are. (The notion of possible v and t will be taken up shortly.)

The komma k is especially useful for notation because it changes value slowly as p changes. More importantly, it represents the difference $\theta - t$, which is a simple one reflecting the capability of representing the fifth harmonic in addition to the Pythagorean capability of representing the third harmonic only. The fifth harmonic allows the just major third, ratio 4:5, while the third harmonic allows only the Pythagorean major third, 64:81. The difference between the two, viz. k, has the ratio 80:81. (For its size in cents, see Chart 1.)

In some circumstances we may need two or more kommata. We may prefer to use d, as it represents the extreme of all t in a temperament (see Chart 1) and is usually positive. We may prefer to use k and q, as one of them is usually positive if both are not zero. We may use k and s, as they are closely related via a congruence needed for determining how many v's there are in a compound t (Rapoport 1993).

Aside from using new signs for kommata, we will occasionally find it useful to subdivide the sharp and flat, which themselves are actually the positive and negative versions of another, unnamed, large komma. But to represent temperaments solely by subdivisions of sharps and flats leads to an excessive dependence on the characteristics of 12-tET and awkward differentiation of intervals in many cases. In some, where the sharp is not expressed at all, i.e. ♯ = ♭ = 0 u., it is clearly pointless to subdivide it.

III. EQUAL TEMPERAMENTS: STRUCTURES AND NOTATIONS

We now turn to notating specific temperaments in ways that reflect their structures. The following is a chart of abbreviations for intervals which will be presented for each ET.

[4] With higher values of a, it is usually easier first to calculate $a \pmod{12}$, then multiply by +5 or –7, and finally apply mod 12 to the result.

abbr.	name	size
Πw	Pythagorean whole tone	$2v - a$
Πh	Pythagorean diatonic semitone (leimma)	$3a - 5v$
♯	Pythagorean chromatic semitone (apotome)	$7v - 4a$
jm3	just minor third	$v - t$
Πm3	Pythagorean minor third	$v - \theta$
jh	just diatonic semitone	$a - v - t$
jc	just chromatic semitone (minor khroma)	$2t - v$

CHART 2

We now present various temperaments in the format exemplified by 17-tET below. It is easiest to determine p first (from $p = 5a$ (mod 12) or $p = -7a$ (mod 12)), which then gives us v, from Chart 1. Following that, we choose t as the major third closest to just (ca. 386.314 cents). Charts 1 and 2 give the remaining intervals, small and large.

It is important to recall that in 12-tET and potentially all multiples of it (24-, 36-, 48-tET, etc.), p, d, k, s, and q all disappear; i.e. they are 0 u.

IV. EXPANSION OF PYTHAGOREAN NOTATION

a = 17 (1 u. = 70.588 cents)

symbol	units	cents	symbol	units	cents
p	1	70.588	v	10	705.882
d	2	141.176	t	5	352.941
k	1	70.588	θ	6	423.529
s	0	0.000			
q	1	70.588	jm3	5	352.941
			Πm3	4	282.353
Πw	3	211.765			
Πh	1	70.588	jh	2	141.176
♯	2	141.176	jc	0	0.000

CHART 3

The second column on the left and the right represents the various intervals in units of the temperament. The third does this in cents, accurate to three decimal places. At the top left in bold face is a followed by the size of the unit of the temperament.

Each temperament discussed is represented by one or more notations, in the following format with a scale upwards from D and units numbered upwards from 0, exemplified once again for 17-tET. Enharmonic equivalents, only a few of which are included for each temperament, are read vertically under any unit number.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	0
D	E♭	D♯	E	F	G♭	F♯	G	A♭	G♯	A	B♭	A♯	B	C	D♭	C♯	D
		F♭			E♯							C♭			B♯		
		E♭/			F♯\							B♭/			C♯\		
		E\			F/							B\			C/		

CHART 4

We may notate 17-tET entirely by Pythagorean signs, as shown on the top two lines of notation. These are determined by taking the v of 10 u. and replicating it up and down a number of times from the base pitch, arbitrarily chosen to be D. Whenever a number of units results which is greater than 16 or less than 0, then 17 is respectively subtracted from or added to the result so that the units number is "in range." For example, if A is 10 u., then E is 10 + 10 u., or 20 u. Because this exceeds 16 u., we subtract 17 from 20 to get 3 u. for E. Similarly, v lower than D, i.e. G, is –10 u., which, after adding 17 u., becomes 7 u.

It soon becomes easier to replicate major seconds, i.e. D E F♯ G♯ A♯ etc. or D C B♭ A♭ G♭ etc. This gives the same end result as replicating perfect fifths, because the whole notation is still Pythagorean, in which the perfect fifth or fourth generates all the notes, and the interval X to X♯ or X to X♭ (where X is any note) is defined as seven v's up or down respectively.

As in many Pythagorean tunings, θ is quite large and Πm3 quite small when compared to their just or even their 12-tET equivalents. Furthermore, t is notated as a diminished fourth (D up to G♭, G down to D♯, etc., from line 1) or augmented second (D up to E♯, G down to F♭, etc., from line 2). Neither is convenient, as neither looks like a third.

Since $p = k = q = 1$, we may choose one of these kommata to designate one unit, thereby solving this problem. The most likely candidate is k, as in the bottom two lines of notation. It preserves for t an interval from, for example, D up to some kind of F♯—in this case F♯\ (third line, unit 5). In many positive temperaments, i.e. where v is greater than the just perfect fifth (701.955 cents), k is the obvious choice of komma to use for notation, because it uses only two terms, θ and t.

In Chart 4, t and jm3 are indicated above and below D, A, and G in the bottom two lines. In its enharmonic equivalents, this notation therefore shows that these two thirds are one interval, e.g. D up to F♯\ or D up to F/, an identity we also learned from Chart 3 (t = jm3 = 5 u.). This equivalence is hidden by notation which uses only sharps and flats, as in the top two lines. The third in question (5 u.) happens to lie very close to half way between the actual just major third (386.314 cents) and just minor third (315.641 cents). It may therefore be interpreted as either or neither, depending on musical treatment of the temperament. There may actually be two major thirds (5 and 6 u.) and two minor thirds (4 and 5 u.), for a total of three intervals, since two are the same.

Many other enharmonic relations, not shown in this and other charts, merely involve consistent use of the komma signs. A usable scale may combine aspects of all lines of notation.

***a* = 53 (1 u. = 22.642 cents)**

symbol	units	cents	symbol	units	cents
p	1	22.642	*v*	31	701.887
d	2	45.283	*t*	17	384.906
k	1	22.642	*θ*	18	407.547
s	0	0.000			
q	1	22.642	jm3	14	316.981
			Πm3	13	294.340
Πw	9	203.774			
Πh	4	90.566	jh	5	113.208
♯	5	113.208	jc	3	67.925

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
D	C𝄪	B𝄪♯	F𝄫	E♭	D♯	C𝄪♯	G𝄫♭	F♭	E	D𝄪	A𝄫𝄫	G𝄫	F	E♯	D𝄪♯	A𝄫♭
E𝄫/	D/	D//	E♭\	D♯\	E♭/	D♯/	E\\	E\	D𝄪\	E/	E//	F\	E♯\	F/	F//	G♭\
			D♯\\													
			D///													

17	18	19	20
G♭	F♯	E𝄪	B𝄫𝄫
F♯\	G♭/	F♯/	G\\

CHART 5

This ET is well known because it is one of the best such approximations of a 3x5 just intonation. It too may be written solely in Pythagorean signs, as shown on the top line. But the awkwardness of the resulting scale of units is obvious. As in 17-tET and other ETs whose $p = 1$, the *k* sign is useful. Here we need two of them, just as we often need two sharps and two flats (𝄪 and 𝄫) in many temperaments. The second line shows some double *k*'s, again with only a few of the enharmonic relations in this temperament up to note 20. An acceptably notated scale would combine aspects of both lines, as the second line also does not present a reasonable scale by itself. As in 17-tET, jc is represented by ♯\\. It is also ///. The difference between the two ETs in this regard is that we do not need to represent jc at all in 17-tET, because it is 0 u., whereas in 53-tET it is 3 u.

Both lines of notation together, with elaborations as suggested by the notes in the bottom two lines, capture the possible relations in a way in which the sharps and flats alone cannot. For example, if we need a perfect fourth below F/ (note 14, an excellent approximation to the interval 5:6 above D), then we use C/ (note 45), not the Pythagorean notation of this note, which is B♯. Clearly, a perfect fourth F/ down to C/ is preferable to F/ down to B♯. Although with B♯ we could use the perfect fourth higher E♯, D up to E♯ is a poor representation of a minor third, compared with D up to F/.

In some situations, even in 53-tET but certainly in those with a still larger *a*, it is necessary to use higher multiples of *k*, i.e. more than two of them for one note. As writing D/// becomes cumbersome, it may be indicated by D$^{3/}$. If we need a quadruple sharp as well, we have D$^{4♯3/}$ or D$^{2𝄪3/}$, which in 53-tET is the same as F𝄪 and G/.

a = 25 (1 u. = 48.000 cents)

symbol	units	cents	symbol	units	cents
p	5	240.000	v	15	720.000
d	1	48.000	t	8	384.000
k	2	96.000	θ	10	480.000
s	3	144.000			
q	–1	–48.000	jm3	7	336.000
			Πm3	5	240.000
Πw	5	240.000			
Πh	0	0.000	jh	2	96.000
♯	5	240.000	jc	1	48.000

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
D	D♯\\	E♭/	E\	F♭//	E	E♯\\	F/	F♯\	G♭//	G	G♯\\	A♭/	G♯\	A♭//	A	A♯\\
D	D▸	D/	E\	E◂	E	E▸	E/	G\	G◂	G	etc.					
D	D)	D))	E((	E(	E	E)	E))	G((	G(	G	etc.					

CHART 6

The top line is notated with k and $2k$. Since k = 2, however, we may need a sign for ½ k.[5] In line 2, ▸ = ½ k up and ◂ = ½ k down. There are also 1½ k up and down, ▸́ and ◂̀. In this temperament, G▸́ = A\.

Line 3 uses only diesis accidentals, up and down singly,) and (, and doubly,)) and ((. While this may seem an odd approach to notating 25-tET, it preserves the pentatony which is available for every temperament whose a is a mulitple of 5. When a = 25, there are five distinct cycles of v, starting on any five consecutive notes, e.g. D((, D(, D, D), and D)).[6]

The more general case for the size of ▸ in units is ▸ = 1/k, where k is greater than or equal to 2 units. In other words, ▸, which is always 1 u., is in some tunings ½ k, in others ⅓ k, etc. In the latter case, ▸́ = 1⅓ k, not 1½ k. These meanings of the various k signs are needed in 54-tET, where k = 3. This is apparent from Chart 8, to be presented shortly. But in order to understand it, we need Chart 7 first.

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[5]From this point, various signs are introduced to represent wholes or parts of k, d, s, q, and p, as well as parts of ♯/♭. The signs for parts of ♯/♭ are already in common use; the others are not. Although the shapes of new signs always raise questions, they will not be discussed much here. From an extensive examination of the literature on the subject, I have deliberately created these symbols to look as they do in every detail, to the extent of creating a font (type character set) which includes them among its more than 100 symbols for microtonal accidentals. Even if handwritten forms of some may be inferred to be awkward, there are simpler equivalents which should prove readily usable. It hardly needs adding that the shapes which have not had any historical basis or development are the most subject to discussion and possible change.

[6]For a fuller discussion of the relatively unknown 25-tET, see Rapoport 1995.

a = 31 (1 u. = 38.710 cents)

symbol	units	cents	symbol	units	cents
p	–1	–38.710	*v*	18	696.774
d	1	38.710	*t*	10	387.097
k	0	0.000	*θ*	10	387.097
s	–1	–38.710			
q	1	38.710	jm3	8	309.677
			Πm3	8	309.677
Πw	5	193.548			
Πh	3	116.129	jh	3	116.129
♯	2	77.419	jc	2	77.419

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
D	D𝄲	D♯	E♭	E𝄳	E	F♭	E♯	F	F𝄲	F♯	G♭	G𝄳	G	G𝄲	G♯	A♭
						E𝄲	F𝄳									
D	D)	D♯	E♭	E(	E	E)	F(	F	etc.							

CHART 7

The longstanding notation for this temperament uses 𝄲 for ½ ♯ and 𝄳 for ½ ♭, as indicated in line 1, as well as 𝄰 and 𝄭 for 1½ ♯ and 1½ ♭ respectively. There seems little point in changing this quite reasonable notation, since it is consistent with the principles discussed in this article. However, we might equally use the *d* signs, since this temperament has a diesis which is very close to the one in just intonation, ratio 125:128. This notational possibility is indicated in line 3. Although for 31-tET it matters little whether semi-sharps and -flats are used or dieseis are used, it is important not to associate generally the divisions of ♯ and ♭ with *d*, as they coincide only in some temperaments.

a = 54 (1 u. = 22.222 cents)

symbol	units	cents	symbol	units	cents
p	6	133.333	*v*	32	711.111
d	3	66.667	*t*	17	377.778
k	3	66.667	*θ*	20	444.444
s	3	66.667			
q	0	0.000	jm3	15	333.333
			Πm3	12	266.667
Πw	10	222.222			
Πh	2	44.444	jh	5	111.111
♯	8	177.778	jc	2	44.444

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
D	D‡	E♭	D^{3}‡	D^{4}‡	E^{5}𝄳	E^{4}𝄳	E^{3}𝄳	D♯	E𝄳	E	E‡	F	F‡	G♭	F^{3}‡	F^{4}‡
	D▸	D▸▸	D/		E♭/		E\	E◂◂	E◂		E▸	E▸▸	E/	E▸	E▸▸	
		D♯\\		E\\				E♭//	F\					F♯\\	F/	

CHART 8

By generalizing ‡ and 𝄳 to mean not ½ ♯ and ½ ♭ (as in 31-tET) but $1/n$ ♯ and $1/n$ ♭ (with n at least 2 and often larger), we arrive at line 1, which uses only sharps and flats and divisions of these into eight parts. It is not very helpful, as it uses too many numerals and obscures the kommata. The k is needed to show where t is, but as k = 3 u., we also need ⅓ k (▸), and perhaps ⅔ k (2 ▸s placed together: ▸▸). We could dispense with many of the divisions of sharps and flats in line 1: see line 2.

In line 2, no numerals are used to show that ▸ = ⅓ k and ▸▸ = ⅔ k, rather than ½ k and 2/2 k respectively. (For 2/2 k, we would of course use the sign for the full syntonic komma, ascending or descending.) Presumably, the use of 3 as denominator would be indicated in the music at the beginning of a work or section in this temperament having these properties, i.e. having this p and this d. This would avoid needless repetition of the numeral 3. If not, it is possible to write, for example, G◂^{3}, where the numeral to the right of the accidental indicates how many signs make up 1/1 of the basic sign, which in this case is a descending k, or \. G◂◂^{3} indicates ⅔ k down from G, meaning the same as G^{2}◂^{3}. Thus, numerals to the left of an accidental indicate a numerator, and to the right a denominator. An absence of a numerator implies one of a sign (e.g. ◂ or ◂^{3}).

There is a need for some separation sign when two accidentals are used together, each of which has a numerator and at least the one on the left has a denominator, e.g. G^{2}‡^{8}, ^{2}▸^{3}: G raised by 2/8 ♯ plus ⅔ k. Although such a combination of signs is unlikely in this temperament, because the note in question is enharmonically simply A\\, it may well occur in some with larger a's. But if such is used here, it is preferable to specify the denominators elsewhere so as to write—for the present example—only G^{2}‡^{2}▸. That is complex enough![7]

[7]Many readers may reject the complex accidental expressions given here. It remains to restate that they are logically formed and not as complex as they may appear. More importantly, it remains to determine, in music using a high a, whether they are necessary or useful.

V. WHEN v LIES OUTSIDE AN ACCEPTABLE RANGE

a = 13 (1 u. = 92.308 cents)

symbol	units	cents	symbol	units	cents
p	5	461.538	v	8	738.462
d	1	92.308	t	4	369.231
k	2	184.615	θ	6	553.846
s	3	276.923			
q	–1	–92.308	jm3	4	369.231
			Πm3	2	184.615
Πw	3	276.923			
Πh	–1	–92.308	jh	1	92.308
♯	4	369.231	jc	0	0.000

0	1	2	3	4	5	6	7	8	9	10	11	12	0
D	C♯	F	E	D♯	G	F♯	B♭	A	D♭	C	B	E♭	D
	G♭			A♭					G♯			A♯	

CHART 9

To discuss this temperament, we acknowledge a limit of possibility or recognizability for v. It has been shown (Blackwood 1985, Rasch 1985) that when $v > 3a/5$, then Πh < 0, i.e. the ascending Pythagorean diatonic semitone, e.g. D up to E♭, actually descends; and similarly, when $v < 4a/7$, then ♯ < 0, i.e. the ascending Pythagorean chromatic semitone, e.g. D up to D♯, actually descends. If we admit the possibility of both Πh and ♯ being 0, then we have a range of recognizability for v: $4a/7 \leq v \leq 3a/5$. Blackwood (1985) suggests that perceptually the range may be slightly narrower, although his music shows that it may also be wider (Blackwood 1980). Even if this point is not resolved, it is convenient to adopt the stated range of recognizability to illustrate what happens when v falls outside it.

In the above disposition of 13-tET, v is unrecognizable, being more than $3a/5$. Πh is negative, causing this normally ascending (and permissibly stationary) interval to descend. The result gives the absurd relationships shown, in which, for example, an ascending diminished fourth (e.g. D up to G♭), is the same as a descending minor second, which is written as if it ascends. As a notation, line 1, whether combined with line 2 or not, is unusable.

a = 13 (1 u. = 92.308 cents)

symbol	units	cents	symbol	units	cents
p	−7	−646.154	v	7	646.154
d	1	92.308	t	4	369.231
k	−2	−184.615	θ	2	184.615
s	−5	−461.538			
q	3	276.923	jm3	3	276.923
			Πm3	5	461.538
Πw	1	92.308			
Πh	4	369.231	jh	2	184.615
♯	−3	−276.923	jc	1	92.308

0	1	2	3	4	5	6	7	8	9	10	11	12	0
D	E	F♯	G♯	A♯	F	G	A	B	G♭	A♭	B♭	C	D
			D♭	E♭					C♯	D♯			

CHART 10

In this version of 13-tET, v is unrecognizable for a different reason: it is less than $4a/7$. ♯ is negative, causing this normally ascending (and permissibly stationary) interval to descend. Needless to add, when an ascending whole tone (e.g. D♭ up to E♭) is the same as a descending augmented third written as if ascending, something is seriously amiss.

But it is possible to rewrite 13-tET to take into consideration the unrecognizability of its v, to use it as a temperament without a perfect fifth. We do this by assigning notes to the temperament taken from a multiple of it, most simply from the ET represented by $2a$, in this case 26-tET. This is shown in Chart 11, where › = ½ d and ‹ = ½ $-d$.

a = 26 (1 u. = 46.154 cents)

symbol	units	cents	symbol	units	cents
p	−2	−92.308	v	15	692.308
d	2	92.308	t	8	369.231
k	0	0.000	θ	8	369.231
s	−2	−92.308			
q	2	92.308	jm3	7	323.077
			Πm3	7	323.077
Πw	4	184.615			
Πh	3	138.462	jh	3	138.462
♯	1	46.154	jc	1	46.154

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
D	D♯	E𝄫	E♭	E	E♯	F♭	F	F♯	F𝄪	G♭	G	G♯	A𝄫	A♭	A	A♯
D	D›	D)	E‹	E	E›	F‹	etc.									

CHART 11

To rewrite 13-tET, we take every second note of 26-tET. The option of removing the 𝄪 and 𝄫 signs is represented in lines 3 and 4 of Chart 12's notation, where *d* is indicated (*q* being equally feasible in both Chart 11 and Chart 12). 13-tET in this disposition no longer has the interval sizes listed in Chart 9 or 10, but those in Chart 12. The fractional intervals are given no cents value because they are not present in the tuning.

a **= 13 (1 u. = 92.308 cents)**

symbol	units	cents
p	–1	–92.308
d	1	92.308
k	0	0.000
s	–1	–92.308
q	1	92.308
Πw	2	184.615
Πh	1.5	
♯	0.5	

symbol	units	cents
v	7.5	
t	4	369.231
θ	4	369.231
jm3	3.5	
Πm3	3.5	
jh	1.5	
jc	0.5	

0	1	2	3	4	5	6	7	8	9	10	11	12	0
D	E𝄫	E	F♭	F♯	G♭	G♯	A♭	A♯	B♭	B♯	C	C𝄪	D
	D𝄪											D𝄫	
	D)											D(	
	E(											C)	

CHART 12

Every note in this 13-tET is half the number of units of the corresponding item in 26-tET: e.g. A♭ is 14 u. in Chart 11, but 7 u. in Chart 12. Among other intervals, *v* is missing entirely. That does not prevent use of 7 or 8 u. as *v*, although Chart 12 does imply that there is no *v* available in the recognizable range.

We could also notate this temperament by taking every third note of 39-tET, as in Chart 13, where 0̂ = *s* and Ϙ = –*s*.

a **= 13 (1 u. = 92.308 cents)**

symbol	units	cents
p	1	92.308
d	0	0.000
k	0.333	
s	0.667	
q	–0.333	
Πw	2.333	
Πh	0.667	
♯	1.667	

symbol	units	cents
v	7.667	
t	4.333	
θ	4.667	
jm3	3.333	
Πm3	3	276.923
jh	1	92.308
jc	1	92.308

0	1	2	3	4	5	6	7	8	9	10	11	12	0
D	E♭/	E\	F	E♯	G\	A♭	G♯	A/	C♭	B	C/	C♯\	D
	D♯Q			F♯Q					B♭ô			D♭ô	

CHART 13

The use of the skhisma sign gives unit 4 the possibility of looking like a major third instead of an augmented 2nd. It could equally be F♯\\. In the latter representation it is lower than t by ⅓ u., which is in turn lower than θ (i.e. F♯) by ⅓ u. Since k = ⅓ u., two k signs down are needed to reach unit 4 from unit 4.667.

Notably, even if kommata and scale degrees are lacking in an ET, they may be used in combination, as long as the resulting pitch is an integral number of units in the tuning. Although not so obvious in Chart 12, the same principle holds there, e.g. in the use of ♯ and ♭.

VI. WHEN A KOMMA IS NEGATIVE

***a* = 33 (1 u. = 36.364 cents)**

symbol	units	cents	symbol	units	cents
p	−3	−109.091	v	19	690.909
d	0	0.000	t	11	400.000
k	−1	−36.364	θ	10	363.636
s	−2	−72.727			
q	1	36.364	jm3	8	290.909
			Πm3	9	327.273
Πw	5	181.818			
Πh	4	145.455	jh	3	109.091
♯	1	36.364	jc	3	109.091

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
D	D♯	D𝄪	E𝄫	E♭	E	E♯	E𝄪	F♭	F	F♯	F𝄪	G𝄫	G♭	G	G♯	G𝄪
			E♭⌣			E⌢		F⌣			F♯⌢	G♭⌣				A♭⌣⌣
			D♯⌢⌢								G♭⌣⌣					

CHART 14

Although this temperament is easy to notate with no more than the usual Pythagorean signs, in so doing we are left with a t up from D being F𝄪. Since k is negative, we have two immediate options besides the one mentioned. The first allows the sign for the ascending syntonic komma to be the one usually indicating descent, and the sign for the descending syntonic komma to be the one usually indicating ascent. As with Charts 9 and 10, this is to be avoided if possible. In the present instance, F♯\ (i.e. k lower than F♯) would have to indicate a note higher than F♯. The second option uses a different komma. Here q is positive, therefore available. It is used in line 2, where ⌢ = q and ⌣ = $-q$. In this temperament, jc is ♯⌢⌢.

There is also a less obvious option in some cases: the use of a d which is not best d but is still acceptable. In the case of 33-tET, the next higher d is positive, giving $t = \theta = 10$ u., $k = 0$, and $d = q = 3$. This d may be used for notation in the normal manner, with signs for ascent and descent signifying ascending and descending dieseis respectively. But because this t is worse than the t of 11 units, the original solution for 33-tET outlined in Chart 14 may be preferable.

***a* = 50 (1 u. = 24.000 cents)**

symbol	units	cents	symbol	units	cents
p	−2	−48.000	v	29	696.000
d	−1	−24.000	t	17	408.000
k	−1	−24.000	θ	16	384.000
s	−1	−24.000			
q	0	0.000	jm3	12	288.000
			Πm3	13	312.000
Πw	8	192.000			
Πh	5	120.000	jh	4	96.000
♯	3	72.000	jc	5	120.000

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
D	D‡	D‡‡	D♯	E𝄬	E♭	E𝄳𝄳	E𝄳	E	E‡	F♭	E♯	F𝄳	F	F‡	F‡‡	F♯
D	D\	D♯/	D♯	E♭/	E♭	E♭\	E/	E	E\	F♭	E♯	F/	F	F\	F♯/	F♯
					D♯\\											

17	18	19	20
F♯	G♭	G𝄳𝄳	G𝄳
F♯\	G♭	G♭\	G/

CHART 15

Although the best d for this temperament is 2, $d = -1$ is imaginable. With this d, it is difficult to notate 50-tET according to the principles stated and illustrated so far, because in it none of the common kommata are positive. This is the result of some bizarre musical circumstances, to be mentioned in Section VII below.

As a increases in size for $p = -2$, $d = -1$, it becomes more difficult to notate a temperament with only Pythagorean signs, because the distance between a note and its sharp or flat becomes large (e.g. in 122-tET, where ♯ = 9 u.). In 50-tET it may be achieved using signs for ⅓ ♯ (e.g. note 1) and 1⅓ ♯ (e.g. note 17) and the equivalent ♭s, as shown in line 1. Ignoring all the kommata, however, is often not a good solution. Descriptive adequacy may be achieved, but explanatory adequacy may suffer. In other words, line 1 covers all the notes, but does it reveal the intervallic structures the best?

There are two further alternatives for this ET: 1) as in option 1 discussed for 33-tET, allow the sign for an ascending komma to signify descent, and the one for a descending komma to signify ascent; or 2) use a different d and t. In 50-tET, as was pointed out, $d = 2$ is better than $d = -1$.

More generally, that none of the five kommata is positive and four are negative suggests that something is quite wrong. Indeed: in this case, t is higher than 400 cents, in fact very close to the true θ, 407.820 cents; and θ, lower than 400 cents, is close to the true t, 386.314 cents. This suggests further that this disposition of 50-tET is invalid, in the sense that t is out of some useful range. It is: simply put, it is farther from the true t than θ is. It is clearly better in this ET to have $\theta = t = 0$, with $d = 2$, as was stated initially.

VII. THE CONDITION OF ALL KOMMATA BEING NEGATIVE

The condition which makes all of the kommata negative, a situation only slightly more extreme than that of Chart 15, may be readily specified.[8] Since p and d are usually set first in an ET in order to determine v and t, it is easiest to specify the condition in terms of them.

1) If $\mathbf{2p < d < p/2}$, then all five kommata (p, d, k, s, q) are negative. Proof:

2) In the bold expression above, p is negative, because $2p < p/2$.

3) d is negative, because $p/2$ is negative and d is less than that.

4) k is negative, because of the following:

If $p < 0$, then $\mathbf{3\theta - a < 0}$.
If $d < 0$, then $\mathbf{a - 3t < 0}$.

Adding the two bold expressions immediately above and recalling that $\theta - t = k$, we arrive at:

$3k < 0$, which means $k < 0$.

5) q is negative, because of the following:

If $d < p/2$, then $\mathbf{2d < p}$.

Substituting expressions from Chart 1 for d and p which use a, v, and t, we find that the bold expression immediately above reduces to:

$-4v - 2t + 3a < 0$.

Since the expression to the left of the 0 is the same as q (see Chart 16 below), $q < 0$.

6) s is negative, because of the following:

If $\mathbf{2p < d}$, then

---

[8]For simplicity, we will discuss this condition rather than the one which makes all of the kommata non-positive, which includes 0. The non-positive situation will be mentioned at the end of the discussion, including in the next footnote.

substituting expressions from Chart 1 for p and d which use a, v, and t, we find that the bold expression immediately above reduces to:

$8v + t - 5a < 0$.

Since the expression to the left of the 0 is the same as s (see Chart 16 below), $s < 0$.

In addition, it is easy to prove that the condition $2p < d < p/2$ is not only sufficient but necessary for all five kommata to be negative. If d goes outside the range indicated, either q or s becomes 0 or positive. In Chart 15, $2d = p$, causing $q = 0$. If $2p = d$, then $s = 0$.[9]

name	symbol	size using v, t, a	example	in Chart 17
syntonic komma	k	$4v - t - 2a$	C up to C/	1-2, 2-4, 4-5
Pythagorean komma	p	$12v - 7a$	C up to B♯	3-5
diesis	d	$a - 3t$	B♯\\\ up to C	1-3
skhisma	s	$8v + t - 5a$	C up to B♯\	3-4
diaskhisma	q	$-4v - 2t + 3a$	B♯\\ up to C	2-3

CHART 16

Chart 16 presents the five kommata once again, this time with their intervals exemplified as upwards to or from C, using only Pythagorean signs and k. It may be summarized by Chart 17. The lowest note is on the left.

1	2	3	4	5
B♯\\\	B♯\\	C	B♯\	B♯

CHART 17

But when all of the intervals are negative, *up to* is replaced throughout by *down to.* In that case, Chart 16 may be summarized by Chart 18, which is Chart 17 backwards. In Chart 18, again the lowest note is on the left.

---

[9]Chart 15 may be covered by the condition $2p \leq d \leq p/2$, which allows for $q = 0$ or $s = 0$. If both are 0, then all five are 0. If all are 0, we have the familiar situation where $a = 12$, 24, or 36. These ETs use only sharps and flats plus divisions and multiples of them. They pose no problem in notation other than the one of keeping track of all the divisions and multiples of the signs for sharp and flat.

Musically questionable results occur with regard to the major third in situations less radical than when $2p < d < p/2$, as Chart 15 shows. The principal problem is preventing t from being farther from 386.314 cents than θ is, as has already been mentioned in connection with Chart 15. This is a subject for one further comment shortly, and many more in both a previous article (Rapoport 1993) and a subsequent one.

1	2	3	4	5
B♯	B♯\	C	B♯\\	B♯\\\

CHART 18

VIII. WHY p IS NOT NEEDED FOR NOTATION

We may well wonder what happens when $p \geq 0$. It is easy to show that in this case there will always be a positive komma other than p,[10] unless all five are 0 (as for a = 12, 24, 36). When $p = 0$ and $d \neq 0$, one or two of the other three kommata are positive, whether d is positive or negative.

This proves what was stated earlier: that p is not needed for notation. If p is negative, either there is a positive komma or there is not. If there is, we may use it. If there is not, in almost all cases there is something wrong musically with the disposition of the tuning. Recall also that p does not express the main problem in equal temperaments, viz. to reconcile multiples of υ with multiples of t.

It is important, however, to realize that while some notations may appear sensible, they may represent musically nonsensical relationships. The availability of a positive komma is a necessary condition for proper notation (except when all are 0), even if it is not sufficient.

IX. THE ONLY PROBLEMATIC EQUAL TEMPERAMENT WITH A LOW a

Before proceeding to some ETs with fairly high a's, it is worth pointing out that of all the ETs with a recognizable υ, only one resists notation by the principles of this article, viz. 14-tET.

a = 14 (1 u. = 85.714 cents)

symbol	units	cents	symbol	units	cents
p	–2	–171.429	υ	8	685.714
d	–1	–85.714	t	5	428.571
k	–1	–85.714	θ	4	342.857
s	–1	–85.714			
q	0	0.000	jm3	3	257.143
			Πm3	4	342.857
Πw	2	171.429			
Πh	2	171.429	jh	1	85.714
♯	0	0.000	jc	2	171.429

---

[10] I am tempted to ruin a famous quotation by saying that I have discovered a truly remarkable proof (of this theorem) which this article is too small to contain. But the proof is unremarkable, and this article is too *big* to contain it.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	0
D		E		F		G		A		B		C		D
D♯		E♭		F♯		G♯		A♭		B♭		C♯		D♯
D♭		E♯		F♭		G♭		A♯		B♯		C♭		D♭

CHART 19

The problem is the odd-numbered notes. Since ♯ = 0, no divisions of the sharp or flat are possible. Because best p and best d are negative, k is also negative, and because of the specific values of p and d, q and s are not positive. There are therefore no signs available for the missing notes. If we use $d = 2$, the problem disappears. Curiously, t is only slightly closer to the just major third when $d = -1$ (and $t = 5$) than when $d = 2$ (and $t = 4$). But both t's are quite far from that value (i.e. 386.314 cents).

Consequently, if we are willing to admit that there is no good t in this tuning, we may resort to the method applied to 13-tET when we found that there was no good v in it. To notate 14-tET, we may take every other note from 28-tET, every third note from 42-tET, etc. The simplest solution, using 28-tET (Chart 20), attaches a diaskhisma to each note, so that the scale for 14-tET reads D, D⌢ or E⌣, E, E⌢ or F⌣, F, etc. There is no t, as it is 4.5 units, although θ remains 4 units.

a = 28 (1 u. = 42.857 cents)

symbol	units	cents	symbol	units	cents
p	–4	–171.429	v	16	685.714
d	1	42.857	t	9	385.714
k	–1	–42.857	θ	8	342.857
s	–3	–128.571			
q	2	85.714	jm3	7	300.000
			Πm3	8	342.857
Πw	4	171.429			
Πh	4	171.429	jh	3	128.571
♯	0	0.000	jc	2	85.714

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
D	D)	D⌢	E(	E	E)	E⌢		F♯		F⌢		G♯		G⌢		A♯
D♭		E⌣		E♭		F⌣	F(	F	F)	G⌣	G(	G	G)	A⌣	A(	A

CHART 20

X. EQUAL TEMPERAMENTS WITH LARGE *a*'s

a = 72 (1 u. = 16.667 cents)

symbol	units	cents	symbol	units	cents
p	0	0.000	v	42	700.000
d	3	50.000	t	23	383.333
k	1	16.667	θ	24	400.000
s	−1	−16.667			
q	2	33.333	jm3	19	316.667
			Πm3	18	300.000
Πw	12	200.000			
Πh	6	100.000	jh	7	116.667
♯	6	100.000	jc	4	66.667

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
D	D/	D//	D³‡	E♭\\	E♭\	D♯	D♯/	D♯//	E³♭	E\\	E\	E	E/	E//	E³‡	F\\
	D‡	D⌢		E♭⌣		E♭		D♯⌢		E⌣	E♭		E‡	E⌢	F³♭	F⌣
				D♯⌣												

CHART 21

For a = 72, 58 u. give a good approximation of the interval 4:7, 33 u. of 8:11, and 50 u. of 8:13. It may be desirable, therefore, to use relatively few accidentals for these positions and show their similarity in notation to the notation for these intervals in just tunings. We have the following: 58 u. = C⌣ or C/(; 33 u. = G³‡ or A♭³♭ (or A♭♭♭♭, A♭♭♭, or A♭♭♭),[11] but using d, 33 u. = G) or A♭(; 50 u. = B♭⌢ or B♭// or B\(. Similarly for intervals relating to any pitch, not just D: the many enharmonic relations may be used to simplify some accidental signs, depending on which note is considered primary at any given time. Other notations for this ET may be found in Sims 1989 and Richter Herf 1979, although they are less clear and less flexible.[12]

As shown earlier, the sign ♭♭ is used to mean 1½ ♭ in 31-tET. In the above example its meaning is generalized to ♭ + ♭, where ♭ is always defined as −1 u. if ♭ < 0 u. Therefore, in the previous paragraph, A♭³♭ (A 9/6 flat) is equivalent to A♭♭♭♭, A♭♭♭, or A♭♭♭. The sign ♭♭♭ means ♭ + ♭♭.

[11]For the interpretation of this note's accidentals, see the next paragraph.

[12]The notations devised by Ezra Sims and Franz Richter Herf cannot be used for many other ETs, because their inventors had no need to consider d, q, and k in 72-tET, and because they have no signs for fractions or multiples of these intervals. Because of this last deficiency in Sims' and Richter Herf's notation, most pitches in their notation of this ET have only one spelling, and many diatonic and enharmonic relations are obscured. That is less of a problem, however, if one uses 72-tET as if it were a just intonation, in which there are no enharmonic equivalents.

***a* = 171 (1 u. = 7.018 cents)**

symbol	units	cents
p	3	21.053
d	6	42.105
k	3	21.053
s	0	0.000
q	3	21.053
Πw	29	203.509
Πh	13	91.228
♯	16	112.281

symbol	units	cents
v	100	701.754
t	55	385.965
θ	58	407.018
jm3	45	315.789
Πm3	42	294.737
jh	16	112.281
jc	10	70.175

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
D	D▸	D▸▸	C𝄪	D´▸	D´▸▸	D)	E♭(	E♭(▸	C𝄪)	D♯(	E♭◂◂	D))	E♭	E♭▸	D♯◂	D♯
D	C𝄪◂◂	C𝄪◂	D/	D‡/	E𝄫(	D)	D‡)	E♭(▸	E𝄫\	D♯(	C𝄪8‡	E𝄫	D♯\	E♭▸	D♯(/	E♭/

17	18	19	20	21	22	23	24	25	26	27	28	29
E((	D♯▸▸	E♭)	F♭(	D♯)◂	D♯)	E(	Eˋ◂◂	Eˋ◂	F♭	E◂◂	E◂	E
D♯	E𝄫)	E♭)	E(\	D♯)◂	E♭(	D♯)	F♭◂◂	E♭))	E\	E‡\	F𝄫/	E

CHART 22

Although it may seem an odd choice, 171-tET has commanded the attention of several authors, e.g. Martin Vogel (1975). All of the signs, representing various fractions of kommata, are explained in Chart 24. Line 1 of Chart 22 shows a notation based on ♯ and *k*, plus the convenience of *d* = 2*k*. The notation on this line is symmetrical about the midpoint, 14½ u. Line 2 shows a notation based on the same kommata, but taking into account various just ratios based on harmonics 3, 5, 7, 11, and 13. Although it could be made symmetrical about the same point, this notation is not, being based on the somewhat arbitrarily chosen intervals in Chart 23.

That line 2 would be hard to use under any circumstances is obvious. Nonetheless, it represents principles which may aid in deriving a usable notation.

ratio	units	note	enh. abbrev.
143:144	1	C♯))/◀	C𝄪◀◀
99:100	2	C𝄪)\\◀	C𝄪◀
80:81	3	D/	
63:64	4	D‡/	
48:49	5	E♭𝄳𝄳\\	E♭𝄳𝄳(
39:40	6	D)	
35:36	7	D‡//	D‡)
32:33	8	E♭(▶	
27:28	9	E♭𝄳\	
24:25	10	D♯\\	D♯(
143:150	11	C𝄪))\◀	C𝄪8‡
20:21	12	E♭𝄳	
128:135	13	D♯\	
52:55	14	E♭▶	

(continued at top of next column)

ratio	units	note	enh. abbrev.
49:52	15	D♯‡(/	
15:16	16	E♭/	
14:15	17	D♯	
13:14	18	E♭𝄳)	
25:27	19	E♭//	E♭)
12:13	20	E(\	
11:12	21	D♯)◀	
32:35	22	E𝄳\\	E𝄳(
91:100	23	D♯)	
10:11	24	F♭(/▶	F♭◀◀
65:72	25	E♭)//	E♭))
9:10	26	E\	
35:39	27	E‡(/	E‡\
25:28	28	F♭𝄳/	
8:9	29	E	

CHART 23

XI. CHART OF ALL THE REQUIRED SIGNS

The complete list of signs usable in notation of equal temperaments is presented in Chart 24. In the chart, n is the number of units in the komma in question. The Pythagorean komma (p), although unnecessary for this kind of notation, is included at the end.[13]

basic symbol	name	1/n up	1/n down	1 up	1 down	1 + 1/n up	1 + 1/n down
♯/♭	sharp/flat	‡	𝄳	♯	♭	♯‡	♭𝄳
		2/n up	**2/n down**	**2 up**	**2 down**	**1 + 2/n up**	**1 + 2/n down**
		‡‡	𝄳𝄳	𝄪	𝄫	♯‡‡	♭𝄳𝄳

basic symbol	name	1/n up	1/n down	1 up	1 down	1 + 1/n up	1 + 1/n down
d	diesis (pl. dieseis)	▸	◂	)	(	)▸	(◂
		2/n up	**2/n down**	**2 up**	**2 down**	**1 + 2/n up**	**1 + 2/n down**
		▸▸	◂◂	))	((	)▸▸	(◂◂

[13]The signs for 1 + 1/n and 1 + 2/n, for example with the basic signs ♯ and ♭, are not very useful when n is very large and a is large (e.g. in 171-tET).

basic symbol	name	1/*n* up	1/*n* down	1 up	1 down	1 + 1/*n* up	1 + 1/*n* down
k	syntonic komma	▸	◂	/	\	/▸	\◂
	(pl. kommata)	2/*n* up	2/*n* down	2 up	2 down	1 + 2/*n* up	1 + 2/*n* down
		▸▸	◂◂	//	\\	/▸▸	\◂◂

basic symbol	name	1/*n* up	1/*n* down	1 up	1 down	1 + 1/*n* up	1 + 1/*n* down
s	skhisma						
	(pl. skhismata)	2/*n* up	2/*n* down	2 up	2 down	1 + 2/*n* up	1 + 2/*n* down

basic symbol	name	1/*n* up	1/*n* down	1 up	1 down	1 + 1/*n* up	1 + 1/*n* down
q	diaskhisma (pl.	⌢̣	⌣̣	⌢	⌣	⌢⌢̣	⌣⌣̣
	diaskhismata)	2/*n* up	2/*n* down	2 up	2 down	1 + 2/*n* up	1 + 2/*n* down
		⌢̣⌢̣	⌣̣⌣̣	⌢⌢	⌣⌣	⌢⌢̣⌢̣	⌣⌣̣⌣̣

basic symbol	name	1/*n* up	1/*n* down	1 up	1 down	1 + 1/*n* up	1 + 1/*n* down
p	Pythagorean komma	▲	▼	^	ˇ	^▲	ˇ▼
	(pl. kommata)	2/*n* up	2/*n* down	2 up	2 down	1 + 2/*n* up	1 + 2/*n* down
		▲▲	▼▼	^^	ˇˇ	^▲▲	ˇ▼▼

CHART 24

XII. CONCLUSION

This article leads to the speculation that the most important komma for notation is *k*, followed in order by *d*, *q*, and *s*. Further study could elucidate this order or challenge it.

The use of any of the above signs may not completely mitigate the complications inherent in this manner of notating close approximations to just tunings for large *a*'s. More concise signs are needed which represent the higher harmonics better, and not necessarily from a Pythagorean starting point. That is another topic. In any case, it is now possible to derive a clear and consistent notation of nearly any ET in terms of its representation of harmonics 3 and 5, and thus of any tonal or modal music (broadly interpreted) which may be written in a wide variety of temperaments. Choices remain, e.g. how much to use subdivisions of the sharp and flat, which kommata to notate in any given situation, which signs to use (if the signs used here are deemed unsuitable), and the order in which to place them on the staff. But whatever the signs, the choices should be made within a comprehensive framework.

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